Periodic solutions of a resistive model for nonlocal Josephson dynamics

2009 J. Phys. A: Math. Theor. 42 025401

(http://iopscience.iop.org/1751-8121/42/2/025401)

J. Phys. A: Math. Theor. **42** (2009) 025401 (7pp) doi:10.1088/1751-8113/42/2/025401

Periodic solutions of a resistive model for nonlocal Josephson dynamics

Yoshimasa Matsuno

Division of Applied Mathematical Science, Graduate School of Science and Engineering, Yamaguchi University, Ube 755-8611, Japan

E-mail: matsuno@yamaguchi-u.ac.jp

Received 25 September 2008, in final form 27 October 2008
Published 25 November 2008
Online at stacks.iop.org/JPhysA/42/025401

Abstract
A novel method is developed for constructing periodic solutions of a model equation describing nonlocal Josephson electrodynamics. This method consists of reducing the equation to a system of linear ordinary differential equations through a sequence of nonlinear transformations. The periodic solutions are then obtained by a standard procedure which is represented in terms of trigonometric functions. It is found that the large time asymptotic of the solution exhibits a steady profile which does not depend on initial conditions.

PACS numbers: 02.30.Ik, 05.45.Yv, 74.50.+r

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The recent studies on Josephson tunnel junctions with high-temperature superconductors reveal that the nonlocal nature of Josephson electrodynamics becomes dominant when the Josephson penetration depth λ_J is shorter than the London penetration depth λ_L. In particular, if we consider a thin layer between two superconductors, the phase difference $\phi(x,t)$ across the Josephson junction is described by the following model equation [1–5]:

$$\omega_J^{-2}\phi_{tt} + \omega_J^{-2}\eta\phi_t = -\sin\phi + \frac{\lambda_J^2}{\pi\lambda_L}\int_{-\infty}^{\infty} K_0\left(\frac{|x-x'|}{\lambda_L}\right)\phi_{x'x'}(x',t)\,\mathrm{d}x' + \gamma. \tag{1}$$

Here, K_0 is the modified Bessel function of order zero, ω_J is the Josephson plasma frequency, η is a positive parameter inversely proportional to the resistance of a unit area of the tunneling junction, γ is a bias current density across the junction normalized by the Josephson critical current density, and the subscripts t and x' appended to ϕ denote partial differentiation. When the characteristic space scale l of ϕ is extremely large compared with λ_L, the kernel K_0 has an approximate expression $K_0(x) \sim \pi\delta(x)$ where $\delta(x)$ is Dirac's delta function. Then, equation (1) reduces to the perturbed sine-Gordon equation [6]. In the opposite limit $l << \lambda_L$, one can use the asymptotic of the kernel $K_0(|x|) \sim -\ln|x|$. In addition, if we restrict our consideration to the overdamped case $\eta >> 1$ as well as the zero bias current $\gamma = 0$, then unlike the perturbed sine-Gordon equation, equation (1) becomes an integrodifferential (or nonlocal) equation. It can be written in an appropriate dimensionless form as

$$\phi_t = -\sin\phi + H\phi_x, \qquad H\phi_x = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_{x'}(x', t)}{x' - x} \, dx', \tag{2}$$

where H is the Hilbert transform operator. Equation (2) may be termed a resistive model for nonlocal Josephson electrodynamics [7]. Note that equation (2) has been proposed for the first time in searching integrable nonlinear equations with dissipation [8]. The general multikink solutions of the equation (2) have been obtained and their properties have been investigated in detail [8].

In this paper, we report some new results concerning periodic solutions of the equation (2). Specifically, we show that equation (2) can be transformed to a finite-dimensional nonlinear dynamical system through a dependent variable transformation. We then linearize the system of equations to derive a first-order system of *linear* ordinary differential equations (ODEs). Its initial value problem can be solved explicitly to obtain periodic solutions. It is shown that the large time asymototic of the periodic solution relaxes to a steady profile independent of initial conditions.

2. Exact method of solution

2.1. A nonlinear dynamical system

We seek a periodic solution of (2) of the form

$$\phi = \mathrm{i} \ln \frac{f^*}{f}, \qquad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin \beta(x - x_j), \tag{3}$$

where $x_j = x_j(t)$ are complex functions of t whose imaginary parts are all positive, β is a positive parameter, N is an arbitrary positive integer and f^* denotes the complex conjugate expression of f. Using a formula for the Hilbert transform, one has $H\phi_x = -(\ln f^* f)_x$. Substitution of this expression and (3) into (2) gives the following bilinear equation for f and f^*:

$$\mathrm{i}(f_t^* f - f^* f_t) = \frac{\mathrm{i}}{2}(f^2 - f^{*2}) - f_x^* f - f^* f_x. \tag{4}$$

We divide (4) by $f^* f$, substitute f from (3) and then evaluate the residue at $x = x_j$ on both sides to obtain a system of nonlinear ODEs for x_j,

$$\dot{x}_j = -\frac{1}{2\beta} \frac{\prod_{l=1}^{N} \sin \beta(x_j - x_l^*)}{\prod_{\substack{l=1 \\ (l \neq j)}}^{N} \sin \beta(x_j - x_l)} + \mathrm{i}, \qquad j = 1, 2, \ldots, N, \tag{5}$$

where an overdot denotes differentiation with respect to t. Note that a dynamical system corresponding to the multikink solution is derived simply from (5) by taking the limit $\beta \to 0$ [8]. As in the multikink case, it will be demonstrated from (5) that the imaginary part of x_j remains positive if it is positive at an initial time.

Before proceeding, it is convenient to introduce some notations

$$z = \mathrm{e}^{2\mathrm{i}\beta x}, \qquad \xi_j = \mathrm{e}^{2\mathrm{i}\beta x_j}, \qquad \eta_j = \mathrm{e}^{2\mathrm{i}\beta x_j^*}, \qquad j = 1, 2, \ldots, N, \tag{6}$$

$$s_1 = \sum_{j=1}^{N} x_j, \qquad s_2 = \sum_{j<l}^{N} x_j x_l, \qquad \ldots, \qquad s_N = \prod_{j=1}^{N} x_j, \tag{7}$$

$$u_1 = \sum_{j=1}^{N} \xi_j, \qquad u_2 = \sum_{j<l}^{N} \xi_j \xi_l, \qquad \ldots, \qquad u_N = \prod_{j=1}^{N} \xi_j, \tag{8}$$

$$v_1 = \sum_{j=1}^{N} \eta_j, \qquad v_2 = \sum_{j<l}^{N} \eta_j \eta_l, \qquad \ldots, \qquad v_N = \prod_{j=1}^{N} \eta_j, \tag{9}$$

$$t_j = \sum_{l=1}^{N} \xi_l^j, \qquad j = 1, 2, \ldots, N. \tag{10}$$

Here, s_j, u_j and v_j are elementary symmetric functions of x_l, ξ_l and η_l $(l = 1, 2, \ldots, N)$, respectively. In terms of u_j $(j = 1, 2, \ldots, N)$ and s_1, f from (3) can be written as

$$f = \frac{\mathrm{e}^{-\mathrm{i}\beta(Nx-s_1)}}{(2\beta\mathrm{i})^N}(z^N - u_1 z^{N-1} + u_2 z^{N-2} + \cdots + (-1)^N u_N). \tag{11}$$

Thus, u_j $(j = 1, 2, \ldots, N)$ and s_1 determine the function f completely. In the following analysis, we derive a system of equations for u_j. To this end, we find it appropriate to rewrite (5) in terms of ξ_j and η_j as

$$\dot{\xi}_j = -\frac{1}{2}\alpha u_N \frac{\prod_{l=1}^{N}(\xi_j - \eta_l)}{\prod_{\substack{l=1\\(l\neq j)}}^{N}(\xi_j - \xi_l)} - 2\beta\xi_j, \qquad j = 1, 2, \ldots, N, \tag{12}$$

where

$$\alpha = \prod_{j=1}^{N}(\xi_j\eta_j)^{-1/2} = \mathrm{e}^{-\mathrm{i}\beta(s_1+s_1^*)}, \qquad u_N = \prod_{j=1}^{N}\xi_j = \mathrm{e}^{2\mathrm{i}\beta s_1}. \tag{13}$$

Later, it will be shown that α is a constant independent of t and u_N obeys a single nonlinear ODE.

2.2. *Linearization*

Here, we show that the system of nonlinear ODEs (12) can be linearized in terms of the variables u_j defined by (8). We multiply ξ_j^{n-1} on both sides of (12) and sum up with respect to j from 1 to N to obtain

$$\frac{1}{n}\dot{t}_n = -\frac{\alpha}{2}u_N\sum_{s=0}^{n}(-1)^s v_s I_{n-s} - 2\beta t_n, \qquad n = 1, 2, \ldots, N, \tag{14}$$

where I_{n-s} is defined by

$$I_{n-s} = \sum_{j=1}^{N}\frac{\xi_j^{N+n-s-1}}{\prod_{\substack{l=1\\(l\neq j)}}^{N}(\xi_j - \xi_l)}. \tag{15}$$

In deriving (14), we have used the identity

$$I_n = 0, \qquad -N+1 \leqslant n \leqslant -1. \tag{16}$$

The time evolution of u_n follows from (14) with the help of the formulae [9]

$$u_n = \frac{(-1)^{n-1}}{n}\sum_{j=0}^{n-1}(-1)^j u_j t_{n-j}, \qquad 1 \leqslant n \leqslant N, \qquad \sum_{j=0}^{n}(-1)^j u_j I_{n-j} = 0, \qquad n \geqslant 1, \tag{17}$$

where $u_0 = 1$ and $I_0 = 1$. In fact, differentiating the first formula in (17) by t and substituting (14) for $\dot{t}_{n-j}$, we can show that the quantity h_n defined by

$$h_n = \dot{u}_n + \frac{\alpha}{2}u_N u_n - \frac{\alpha^{-1}}{2}u_{N-n}^* + 2\beta n u_n, \qquad n = 1, 2, \ldots, N, \tag{18}$$

satisfies the relation

$$h_n = \frac{(-1)^{n-1}}{n}\sum_{j=0}^{n-1}(-1)^j h_j t_{n-j} + \frac{(-1)^{n+1}r_n}{2n\alpha}, \tag{19}$$

where

$$r_n = \sum_{j=1}^{n} u_{N-j+n}^* \left[-\sum_{s=1}^{j}(-1)^{n-s} s I_{j-s} + (-1)^{n-j} t_j \right]. \tag{20}$$

A straightforward calculation using (17) shows that the quantity in the brackets on the right-hand side of (20) vanishes identically so that $r_n \equiv 0$. It follows from this and (19) that

$$h_n = \frac{(-1)^{n-1}}{n}\sum_{j=0}^{n-1}(-1)^j h_j t_{n-j}, \qquad n = 1, 2, \ldots, N. \tag{21}$$

Note from (13) and (18) that $h_0 = \alpha u_N/2 - u_N^*/(2\alpha) = 0$ which, combined with (21), leads to the relations $h_n \equiv 0\,(n = 1, 2, \ldots, N)$. Thus, we see that u_n evolves according to the following system of ODEs:

$$\dot{u}_n + \frac{\alpha}{2}u_N u_n - \frac{\alpha^{-1}}{2}u_{N-n}^* + 2\beta n u_n = 0, \qquad n = 1, 2, \ldots, N. \tag{22}$$

It is remarkable that u_N obeys a single nonlinear ODE of the form

$$\dot{u}_N + \frac{\alpha}{2}u_N^2 - \frac{\alpha^{-1}}{2} + 2\beta N u_N = 0, \tag{23}$$

and other $N - 1$ variables $u_1, u_2, \ldots, u_{N-1}$ constitute a system of *linear* ODEs. Substituting (13) into (23), we can put (23) into a nonlinear ODE for s_1,

$$\dot{s}_1 = \frac{1}{2\mathrm{i}\beta}\sinh(2\beta\,\mathrm{Im}\,s_1) + \mathrm{i}N, \tag{24}$$

where $\mathrm{Im}\,s_1$ implies the imaginary part of s_1.

3. Periodic solutions

The first step toward constructing periodic solutions is to integrate (24). It follows from the real and imaginary parts of (24) that

$$\mathrm{Re}\,\dot{s}_1 = 0, \qquad \mathrm{Im}\,\dot{s}_1 = -\frac{1}{2\beta}\sinh(2\beta\,\mathrm{Im}\,s_1) + N. \tag{25}$$

Thus, the real part of s_1 becomes a constant $\mathrm{Re}\, s_1(t) = \mathrm{Re}\, s_1(0) \equiv b$ whereas integration of the equation for $\mathrm{Im}\, s_1$ yields an explicit expression. In terms of a new variable $y = 2\beta\, \mathrm{Im}\, s_1$, it is given by

$$\mathrm{e}^{-y} = \frac{2\nu_N\left(-\tanh\frac{y_0}{2}+1\right)\cosh\nu_N t + \left\{(2\beta N+1)\tanh\frac{y_0}{2} - 2\beta N + 1\right\}\sinh\nu_N t}{2\nu_N\left(\tanh\frac{y_0}{2}+1\right)\cosh\nu_N t + \left\{(2\beta N-1)\tanh\frac{y_0}{2} + 2\beta N + 1\right\}\sinh\nu_N t}, \tag{26}$$

where $\nu_N = \sqrt{(\beta N)^2 + (1/4)}$ and $y_0 = y(0) = 2\beta\, \mathrm{Im}\, s_1(0)$, For $n = 1, 2, \ldots, N-1$, on the other hand, (22) can be written in the form

$$\dot{u}_n = -\left(\frac{1}{2}\,\mathrm{e}^{-2\beta\,\mathrm{Im}\, s_1} + 2\beta n\right) u_n + \frac{\alpha^{-1}}{2} u^*_{N-n}. \tag{27}$$

Note from (13) and $\mathrm{Re}\, s_1 = b$ that $\alpha = \mathrm{e}^{-2\mathrm{i}\beta b}$ becomes a constant. The solution of the initial value problem for (27) can be obtained by means of a standard procedure. It can be put into the form of a rational function

$$u_n(t) = \frac{G_n}{F}, \qquad n = 1, 2, \ldots, N-1, \tag{28}$$

with

$$F = 2\nu_N\left(\tanh\frac{y_0}{2}+1\right)\cosh\nu_N t + \left\{(2\beta N-1)\tanh\frac{y_0}{2} + 2\beta N + 1\right\}\sinh\nu_N t, \tag{29}$$

$$G_n = 2\nu_N\left(\tanh\frac{y_0}{2}+1\right)\left[u_n(0)\cosh\nu_n t + \frac{1}{\nu_n}\left\{\beta(N-2n)u_n(0) + \frac{\alpha^{-1}}{2}u^*_{N-n}(0)\right\}\sinh\nu_n t\right], \tag{30}$$

where $\nu_n = \sqrt{\beta^2(N-2n)^2 + (1/4)}$. We see that expression (28) with $n = N$ produces (26) and hence it can be used for all u_n.

A novel feature of the solution given above will become apparent if one explores the large time asymptotic of the solution. Actually, it is easy to see from (28), (29) and (30) that as t tends to infinity, $u_n (n = 1, 2, \ldots, N)$ approach the following limiting values:

$$u_n \to 0, \qquad n = 1, 2, \ldots, N-1, \qquad u_N \to \mathrm{e}^{2\mathrm{i}\beta b}\left(\sqrt{4(\beta N)^2+1} - 2\beta N\right). \tag{31}$$

The asymptotic form of ϕ follows from (3), (11) and (31), giving rise to

$$\phi \sim 2\tan^{-1}\left[\frac{\sqrt{4(\beta N)^2+1}-1}{2\beta N}\tan\beta\left(Nx - b - \frac{N\pi}{2\beta}\right)\right]. \tag{32}$$

If we introduce a new variable u by $u = \phi_x$, then in the limit $t \to \infty$, u behaves like

$$u \sim \frac{4(\beta N)^2}{\sqrt{4(\beta N)^2+1} + (-1)^N\cos 2\beta(Nx-b)}. \tag{33}$$

It is remarkable that the asymptotic form of u does not depend on initial conditions except for a phase constant b. It represents a train of nonlinear periodic waves with an equal amplitude. Since $u_1(0) \neq 0$, the initial profile of u has a spatial period π/β whereas that corresponding to (33) is given by $\pi/N\beta$. Therefore, as time evolves, there appear N identical waves in the space interval π/β. The maximum and minimum values of each wave are given, respectively, by $u_{\max} = \sqrt{4(\beta N)^2+1} + 1$ and $u_{\min} = \sqrt{4(\beta N)^2+1} - 1$. If we define the amplitude of the wave by $A = u_{\max} - u_{\min}$, then $A = 2$, indicating that the amplitude becomes a constant independent of the wavenumber. Figure 1 shows a typical time evolution of u for $N = 2$ where the parameters are chosen as $\beta = 0.2$, $u_1(0) = \mathrm{e}^{-1.6} + \mathrm{e}^{-0.8}$, $u_2(0) = \mathrm{e}^{-2.4}$, $(x_1(0) = 4\mathrm{i}, x_2(0) = 2\mathrm{i})$. In this example, the wavelength of the periodic wave is 7.85. As expected from the asymptotic form (33), we can observe two identical waves with an amplitude 2 in the space interval 15.7 at a final stage of the time evolution.

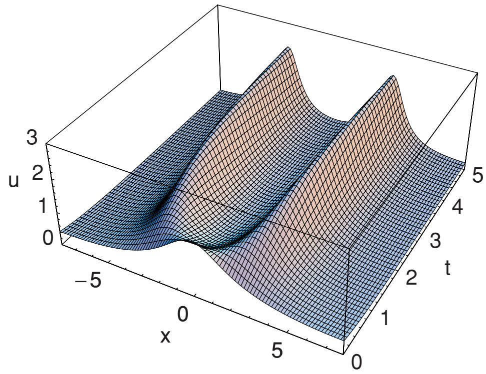

Figure 1. Time evolution of u for $N = 2$.

4. Conclusion

We have developed an exact method for constructing periodic solutions of a resistive model equation for nonlocal Josephson electrodynamics. Although the basic equation (2) is a highly nonlinear equation with a nonlocal term, it can be linearized through a sequence of dependent variable transformations. We have obtained various new results. Among them, a system of equations (22) is a crucial consequence for the purpose of determining the function f given by (11) in a closed form. The large time asymptotic of the solution exhibits a novel feature. In particular, it relaxes to a steady profile whose functional form does not depend on initial conditions except for a phase constant.

We conclude this paper with some comments. First, we point out that for $N = 1$, the periodic solution presented here reproduces an existing solution which has been obtained by a different method [10]. The periodic solutions for $N \geqslant 2$ appear here for the first time. It is interesting to see that the large time asymptotic (32) for ϕ satisfies the static version of (2), i.e., $H\phi_x = \sin\phi$. As already pointed out [8], this equation is a model of dislocation derived by Peierls [11, 12]. Second, we can perform a similar analysis for a resistive model with a bias current γ. It is a relatively easy task to obtain multikink solutions following a procedure developed in [8]. Nevertheless, the construction of periodic solutions deserves further study. Third, the method of solution developed here is applicable to other nonlocal nonlinear evolution equations as well. For example, we will be able to obtain periodic solutions of the sine-Hilbert equation $H\theta_t = -\sin\theta, \theta = \theta(x, t)$. It is noteworthy that the construction of periodic solutions of the sine-Hilbert equation has been done by a different method, but the explicit solutions have been presented only for $N = 1$ [13]. On the other hand, our method will enable us to obtain periodic solutions for general N. Fourth, the periodic solutions can be used to calculate various physical quantities such as the current density and the electric and magnetic fields in a Josephson junction. These quantities can be compared with experimental results for high-temperature Josephson junctions. The solutions to the various problems mentioned above will be reported in a subsequent paper as well as a detailed description of the present short communication.

References

[1] Aliev Yu M, Silin V P and Uryupin S A 1992 *Superconductivity* **5** 230–6
[2] Gurevich A 1992 *Phys. Rev.* B **46** 3187–90
[3] Aliev Yu M and Silin V P 1993 *JETP Lett.* **77** 142–7

[4] Gurevich A 1993 *Phys. Rev.* B **48** 12857–65
[5] Mints R G 1997 *J. Low Temp. Phys.* **106** 183–92
[6] Barone A and Paterno G 1982 *Physics and Application of the Josephson Effect* (New York: Wiley)
[7] Silin V P and Uryupin S A 1995 *JETP Lett.* **81** 1179–91
[8] Matsuno Y 1992 *J. Math. Phys.* **33** 3039–45
[9] Matsuno Y 2004 *J. Math. Phys.* **45** 795–802
[10] Alfimov G L and Silin V P 1994 *JETP Lett.* **79** 369–76
[11] Peierls R 1940 *Proc. Phys. Soc.* **52** 34–7
[12] Nabarro F R N 1947 *Proc. Phys. Soc.* **59** 256–72
[13] Matsuno Y 1987 *Phys. Lett.* A **120** 187–90